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LETTER TO THE EDITOR

S-matrix poles for chaotic quantum systems as eigenvalues of complex symmetric random matrices: from isolated to overlapping resonances

Hans-Jürgen Sommers†, Yan V Fyodorov†‡ and Mikhail Titov‡

† Fachbereich Physik, Universität-GH Essen, D-45117 Essen, Germany
‡ Petersburg Nuclear Physics Institute, Gatchina 188350, Russia

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Abstract. We study complex eigenvalues of large $N \times N$ symmetric random matrices of the form $\mathcal{H} = \hat{H} - \mathrm{i}\hat{\Gamma}$, where both $\hat{H}$ and $\hat{\Gamma}$ are real symmetric, $\hat{H}$ is a random Gaussian and $\hat{\Gamma}$ is such that $N \operatorname{Tr} \hat{\Gamma}^2 \sim \operatorname{Tr} \hat{H}^2$ when $N \to \infty$. When $\hat{\Gamma} \geqslant 0$ the model can be used to describe the universal statistics of S-matrix poles (resonances) in the complex energy plane. We derive the ensuing distribution of the resonance widths which generalizes the well known χ^2 distribution to the case of overlapping resonances. We also consider a different class of 'almost real' matrices when $\hat{\Gamma}$ is random and uncorrelated with $\hat{H}$.

As is well known, S-matrix poles or *resonances* can be regarded as complex eigenvalues of an effective non-hermitian Hamiltonian $\mathcal{H}$ emerging after the so-called *complex rotation* was applied to the original hermitian Hamiltonian of the system. The method [1] allows one to extract resonance positions E_k and widths Γ_k directly, without expensive evaluation of the energy-dependent S-matrix elements. The search of resonances typically amounts to diagonalizing large complex symmetric matrices representing the effective Hamilonian in a suitable basis on a finite grid. Using this method, patterns consisting of hundreds of individual resonances were recently extracted for realistic models in atomic and molecular physics [2, 3]. Frequently, for high enough excitation energies the S-matrix poles are placed irregularly in the complex plane, forming a structure of a 'chaotic jumble' [3], the situation calling for a statistical description. As another important development it is appropriate to mention a powerful numerical algorithm that was proposed to extract resonance positions and widths with a high accuracy from the experimentally accessible time-correlation function of finite duration [4]. All these facts make the problem of statistical description of resonances to be an important and opportune task.

At the same time, the statistics of highly excited *bound* states of closed quantum systems of quite different microscopic nature is known to be system-independent (universal), provided the corresponding classical counterparts demonstrate a well-developed chaotic motion [5]. Moreover, the spectral correlations turn out to be exactly the same as those provided by the theory of large random matrices on the scale determined by a typical separation Δ between neighbouring eigenvalues. Microscopic justifications of the use of random matrices for describing the universal properties of quantum chaotic systems have been provided by several groups recently, based both on traditional semiclassical periodic orbit expansions [6, 7] and

on advanced field-theoretical methods [8, 9]. It is natural to try to develop a description of universal statistical properties of resonances using the same ideas.

The methods to adjust random matrix description to the case of open chaotic systems are well known since the pioneering paper [10] and described in much detail in [11]. The starting point of this approach is the representation of the scattering matrix $\hat{S}$ in terms of an effective non-hermitian Hamiltonian $\mathcal{H}_{eff} = \hat{H} - \mathrm{i}\hat{\Gamma}$. Here the hermitian Hamiltonian $\hat{H}$ describes the *closed* counterpart of the open system whereas the anti-hermitian part $\mathrm{i}\hat{\Gamma}$ arises due to coupling to open scattering channels. It has to be chosen in the form ensuring the unitarity of the scattering matrix. As a result all eigenvalues of $\hat{\Gamma}$ turn out to be non-negative (as is also required by causality), the number of strictly positive eigenvalues being just the number M of open channels.

It is natural to expect that statistical properties of resonances are inherited from the corresponding random matrix universality of levels of closed systems. Remembering the discussion above, we expect them to be universal on the energy scales in the complex plane comparable with the mean level spacing for the closed system Δ. In contrast, properties on a much larger scale must be highly system-specific.

A natural way to incorporate the random matrix description of the quantum chaotic system is to replace $\hat{H}$ by a large $N \times N$ random matrix of appropriate symmetry. Namely, chaotic systems with preserved time-reversal invariance (TRI) should be described by matrices H_{ij} which are real symmetric. Such matrices form the Gaussian orthogonal ensemble (GOE), whereas for systems with broken TRI one uses complex hermitian matrices from the Gaussian unitary ensemble (GUE) [5]. As for the matrix $\hat{\Gamma}$ it is determined by the coupling of the chaotic system to open channels and can be considered as fixed, given one of the same dimension.

In general, the two matrices $\hat{H}$ and $\hat{\Gamma}$ do not commmute, making the analysis of complex eigenvalues of $\mathcal{H}$ to be a rather non-trivial problem. In the simplest case one can treat the anti-hermitian part as a small perturbation to the hermitian one. This is justified for the regime of isolated resonances, i.e. when typical resonance widths are much smaller than the mean separation Δ between the positions of neighbouring resonances. Assuming for simplicity that all scattering channels (hence, the eigenvalues of $\hat{\Gamma}$) are statistically equivalent, one arrives at the so-called χ^2 distribution of the resonance widths. It was originally introduced many years ago by Porter and Thomas and has since then been observed experimentally in many physical systems, see some references in [12]. The corresponding perturbative expression is also known for the case of non-equivalent scattering channels [13].

Until quite recently, relatively little was known about resonance statistics non-perturbatively. Among the known facts were the joint probability density of all resonances (not, however, its moments) for the systems with only one open channel $M = 1$ [14], and the mean density of complex eigenvalues for very many open channels $M \sim N \gg 1$ [15]. The most essential progress was achieved recently [11, 16] by employing a powerful field-theoretical method known as Efetov's supersymmetric nonlinear σ-model [17]. It turns out to be possible to find explicitly the mean density of the resonances for systems with broken time-reversal invariance and to put forward a well-grounded conjecture on the higher-correlation functions in the complex plane [18]. Unfortunately, the most interesting and practically important case of the systems with preserved time-reversal invariance proved to be less accessible analytically (for numerics see [19]) and no results of such generality have been obtained for such systems so far.

The main goal of this letter is to fill in this gap partly by presenting the exact non-pertubative distribution of the resonance widths for quantum chaotic systems with preserved time-reversal invariance.

It is necessary to mention that random matrices (more generally, random linear operators)

with complex eigenvalues emerged recently in many other physical contexts. A few recent examples include the description of thermal motion of an isolated vortex in disordered type-II superconductors with columnar defects, the problem of a classical diffusing particle advected by a stationary random velocity field and attempts to understand the universal features of chiral symmetry breaking in quantum chromodynamics (a more detailed discussion and the relevant references can be found in the papers mentioned below). At the same time, our knowledge of statistical properties of random non-self-adjoint matrices is quite scarce and incomplete. This fact recently stimulated efforts of different groups to improve our understanding in this direction [20–24]. In particular, the existence of a non-trivial regime of *weak non-hermiticity* was recognized in [22], see more detailed discussion in [24]. The guiding idea to realize the existence of such a regime comes from the experience with resonances [16]. Guided by that example one guesses that a new regime occurs when the imaginary part of typical eigenvalues is comparable with the mean *separation* between neighbouring eigenvalues along the real axis.

One can again use the supersymmetry approach to obtain the mean density of complex eigenvalues in the regime of weak non-hermiticity for matrices with independent elements [22, 24]. The density turned out to be described by a formula containing, as two opposite limiting cases, both the Wigner semicircular density of real eigenvalues of the hermitian matrices and the uniform density of complex eigenvalues already discovered for usual non-hermitian random matrices by Ginibre [25], addressed in much detail by Girko [26] and other authors [27]. In his insightful paper Efetov [28] managed to derive the density of complex eigenvalues for a related, but different set of (almost symmetric) real random matrices. His result was generalized in [18]. Actually, the non-hermitian matrices considered in [22] and [28] are just two limiting cases of a general three-parameter family of 'weakly non-hermitian' ensembles [24] with independent entries. Apart from these two cases, the mentioned three-parameter family contains one more important particular case corresponding to an ensemble of complex symmetric random matrices. It is natural to call such matrices 'almost-real' ones. The second goal of this paper is to provide the eigenvalue density for this particular ensemble.

As a starting point of our investigation we use the representations of the mean density $\rho(X, Y)$ of eigenvalues $Z = X + \mathrm{i}Y$ with a given imaginary part Y derived by us earlier [24, 29]. Let us first consider the case relevant to the chaotic scattering. Due to rotational invariance of the GOE matrices the result can be dependent only on the set of positive eigenvalues $\gamma_1, \gamma_2, \ldots, \gamma_M$ of the matrix $\hat{\Gamma}$. Correspondingly, the density $\rho_X(y) \equiv \rho(X, Y)\Delta^2(X)/\pi$ of scaled resonance widths $y = \frac{\pi Y}{\Delta}$ (measured in units of the local mean level spacing $\Delta(X) = 1/(N\nu(X))$ of the closed system, with $\nu(X) = \pi^{-1}(1 - X^2/4)^{1/2}$ being the semicircular density of real eigenvalues) for those resonances whose positions are within the narrow window around the point X of the spectrum is given by:

$$\langle \rho_X(y) \rangle = \frac{1}{16} \int \mathrm{d}\mu\,(\hat{Q})\,\mathrm{Str}(\hat{\sigma}_\tau^{(F)}\hat{Q})\mathrm{Str}(\hat{\sigma}_\tau \hat{Q}) \times \prod_{a=1}^{M} \mathrm{Sdet}^{-1/4}\left[1 - \frac{\mathrm{i}}{2g_a}\{\hat{Q}, \hat{\sigma}_\tau\}\right] \exp\frac{\mathrm{i}}{2} y\,\mathrm{Str}(\hat{\sigma}_\tau \hat{Q}). \tag{1}$$

We introduced here the parameters $g_a \in [1, \infty)$ characterizing the coupling of a system to open channels which are expressed in terms of the eigenvalues γ_a as $g_a = \frac{1}{2\pi\nu(X)}(\gamma_a + \gamma_a^{-1})$. The integration above goes over the set of 8×8 supermatrices $\hat{Q}$ satisfying the constraint $\hat{Q}^2 = -1$ and some symmetry requirements, which can be conveniently resolved if one parametrizes the manifold as $\hat{Q} = -\mathrm{i}Z\Lambda Z^{-1}$, with $\Lambda = \mathrm{diag}(\mathbf{1}, -\mathbf{1})$, each $\mathbf{1}$ representing a 4×4 unity matrix. The symbols Str and Sdet stand for the graded trace and the graded determinant, respectively, and $\{\hat{Q}, \hat{\sigma}_\tau\} = \hat{Q}\hat{\sigma}_\tau + \hat{\sigma}_\tau \hat{Q}$ stands for the anticommutator. Other 8×8 supermatrices entering

this expressions are as follows:

$$\hat{\sigma}_\tau^{(F)} = \begin{pmatrix} \hat{0}_4 & \hat{\tau}_3^{(F)} \\ \hat{\tau}_3^{(F)} & \hat{0}_4 \end{pmatrix} \qquad \hat{\sigma}_\tau = \begin{pmatrix} 0 & \hat{\tau}_3 \\ \hat{\tau}_3 & 0 \end{pmatrix}$$

where $\hat{\tau}_3$ and $\hat{\tau}_3^{(F)}$ are 4×4 diagonal supermatrices: $\hat{\tau}_3 = \mathrm{diag}\{\hat{\tau}, \hat{\tau}\}$; $\hat{\tau}_3^{(F)} = \mathrm{diag}\{\hat{0}_2, \hat{\tau}\}$ with $\hat{\tau} = \mathrm{diag}(1, -1)$.

For the case of the matrix $\hat{\Gamma}$ being real symmetric with entries independently fluctuating around zero and normalized in such a way that $\lim_{N\to\infty} \mathrm{Tr}\, \hat{\Gamma}^2 < \infty$ (the condition of being 'almost symmetric'), the corresponding expression for the mean density of eigenvalues is obtained from equation (1) by replacing the product of superdeterminants in the integrand with the exponent:

$$\exp\left\{-\frac{v^2}{4}\,\mathrm{Str}(\hat{\sigma}_\tau \hat{Q})^2\right\}$$

where $v^2 = (\pi\nu)^2 \lim_{N\to\infty} \mathrm{Tr}\, \hat{\Gamma}^2$, see the derivation in [24].

The starting formulae are actually valid both for real symmetric and complex hermitian matrices H. To extract the explicit form of the distribution function one still has to perform the integration over the manifold of the supermatrices Q. Still, it is a rather difficult calculation due to a cumbersome structure of that manifold, and the success is mainly determined by finding an appropriate parametrization. For the simplest case of matrices $\check{H}$ being complex hermitian, the supermatrices Q become block-diagonal due to additional constraints and the calculation, although cumbersome, can still be done successfully [11, 16, 22] in the 'standard' Efetov parametrization [17]. For the case of real symmetric matrices with a resulting full 8×8 structure of Q, this is no longer the case. Fortunately, we are able to parametrize the matrices Z as $Z_1 Z_2$ in such a way, that Z_1 commutes with $\hat{\sigma}_\tau$, thus dropping out from the integrand. We would like to stress that although we enjoyed some insights from the recent paper on real asymmetric matrices [28], our actual parametrization is quite different from that used in [28]. A derivation of the explicit form of such a parametrization and especially the integration measure turn out to be very lengthy and will be presented in detail elsewhere. Here we would like to present only the result of integrating out the Grassmannian degrees of freedom:

$$\begin{aligned}\langle \rho_X(y)\rangle = \frac{-1}{16\pi}\frac{\partial}{\partial y}\int_{-1}^{1} \mathrm{d}\lambda\, f_s^{-2}(\lambda) \int_{-\infty}^{\infty} \mathrm{d}\lambda_1\, f_s(\mathrm{i}\lambda_1) \\ \times \int_{\lambda_1}^{\infty} \mathrm{d}\lambda_2\, f_s(\mathrm{i}\lambda_2) \frac{(\lambda_2-\lambda_1)(2\lambda - \mathrm{i}\lambda_1 - \mathrm{i}\lambda_2)}{(\lambda-\mathrm{i}\lambda_1)^2(\lambda-\mathrm{i}\lambda_2)^2}\end{aligned} \tag{2}$$

where we use the convention $s = r$ for the case of resonance statistics and $s = R$ for that case when the antihermitian part $\hat{\Gamma}$ is a random matrix. Correspondingly, we defined

$$f_s(\lambda) = \frac{\exp \lambda y}{(1-\lambda^2)^{1/2}} \times \begin{cases} \dfrac{1}{\prod_{a=1}^{M}(g_a - \lambda)^{1/2}} & \text{for resonances, } s = r \\ \exp(v^2\lambda^2) & \text{for a random matrix, } s = R. \end{cases}$$

The following important comment is appropriate here. The integrals over λ_1, λ_2 as they stand in equation (2) are not well defined when taken literally along the real axis due to a singularity at the points $\lambda_1 = \lambda = 0$; $\lambda_2 = \lambda = 0$. This difficulty can be traced back to a singularity of the transformation arising from diagonalization of some supermatrices when coming to a new parametrization as mentioned above. A more accurate consideration shows that the integration over λ_1, λ_2 should be understood in the sense of a principal value. To

be precise, the correct expression is equal to the half-sum of two integrals with integration contours encircling the singular point $\lambda_1 = -\mathrm{i}\lambda$; $\lambda_2 = -\mathrm{i}\lambda$ from above and from below.

The expression (2) is exact, but is still not very convenient for evaluation. Fortunately, for the physically motivated case of resonances one can deform the contour in such a way that it goes along the cut $\lambda_{1,2} = -\mathrm{i}p$, $p \in [1, \infty)$ for negative values of y and along a similar cut $\lambda_{1,2} = \mathrm{i}p$, $p \in [1, \infty)$ for $y > 0$. At the same time, all the singularities $\lambda_1 = -\mathrm{i}g_a$, $a = 1, \ldots, M$, belong to the cut in the lower half-plane. This analytic structure of the integrand ensures vanishing of the density for $y > 0$ as expected, whereas for $y < 0$ one obtains:

$$\begin{aligned}\rho_X(y<0) = \frac{1}{4\pi}\frac{\partial^2}{\partial y^2}\int_{-1}^{1}\mathrm{d}\lambda\,(1-\lambda^2)\int_1^\infty \frac{\mathrm{d}p_1}{(p_1^2-1)^{1/2}} \\ \times\int_1^{p_1}\frac{\mathrm{d}p_2}{(p_2^2-1)^{1/2}}\frac{(p_1-p_2)\mathrm{e}^{+y(p_1+p_2-2\lambda)}}{(\lambda-p_1)^2(\lambda-p_2)^2} \\ \times\chi_1(p_2)\chi_2(p_1)\prod_{a=1}^{M}\frac{(g_a-\lambda)}{(|g_a-p_1||g_a-p_2|)^{1/2}}\end{aligned} \tag{3}$$

where the functions $\chi_{1,2}(p)$ are defined as follows:

$$\chi_1(p) = \begin{cases} 1 & \text{for } g_{4k} \leqslant p \leqslant g_{4k+1} \\ -1 & \text{for } g_{4k+2} \leqslant p \leqslant g_{4k+3} \\ 0 & \text{otherwise} \end{cases}$$

$$\chi_2(p) = \begin{cases} 1 & \text{for } g_{4k+1} \leqslant p \leqslant g_{4k+2} \\ -1 & \text{for } g_{4k+3} \leqslant p \leqslant g_{4k+4} \\ 0 & \text{otherwise} \end{cases}$$

where $k = 0, 1, \ldots, [M/4]$ and we used a convention: $1 \equiv g_0 \leqslant g_1 \leqslant \cdots \leqslant g_M < g_{M+1} = \infty$.

The formulae equations (3) provide us with the most general explicit analytical expression for the density of complex resonances for a chaotic quantum system with preserved time-reversal invariance. As such they constitute the main result of this paper.

We see, that in general these formulae are still quite cumbersome. Even for the case of only one open channel ($M = 1$) the expression is lengthy:

$$\rho_X(y<0) = \frac{1}{4\pi}\frac{\partial^2}{\partial y^2}\int_{-1}^{1}\mathrm{d}\lambda\,(1-\lambda^2)\mathrm{e}^{-2\lambda y}F(\lambda, y) \tag{4}$$

where

$$\begin{aligned}F(\lambda, y) = (g-\lambda)\int_g^\infty \mathrm{d}p_1\,\frac{\mathrm{e}^{yp_1}}{(p_1^2-1)^{1/2}(\lambda-p_1)^2(p_1-g)^{1/2}} \\ \times\int_1^g \mathrm{d}p_2\,\frac{\mathrm{e}^{yp_2}}{(p_2^2-1)^{1/2}}\frac{(p_1-p_2)}{(\lambda-p_2)^2(g-p_2)^{1/2}}.\end{aligned}$$

We show in figures 1 and 2 that the obtained expression agrees favourably with the results of direct numerical diagonalization of the corresponding complex matrices.

The general expression (3) can be simplified in various limiting cases. For example, we can consider the case of weak coupling to continuum when all $g_a \gg 1$. By noticing that the integration over p_2 in equation (3) is dominated in this case by $p_2 \sim 1$, whereas $p_1 \geqslant g_1 \gg 1, \lambda, p_2$ and also taking into account that typical values of y are of the order of $g_1^{-1} \ll 1$, one can evaluate the integrals over p_2, λ to the leading order. The result is

$$\rho_X(y<0) = \frac{1}{\pi}\prod_a g_a^{1/2}\int_1^\infty \frac{\mathrm{d}p_1}{\prod_a |g_a-p_1|^{1/2}}\mathrm{e}^{yp_1}\chi_2(p_1) \tag{5}$$

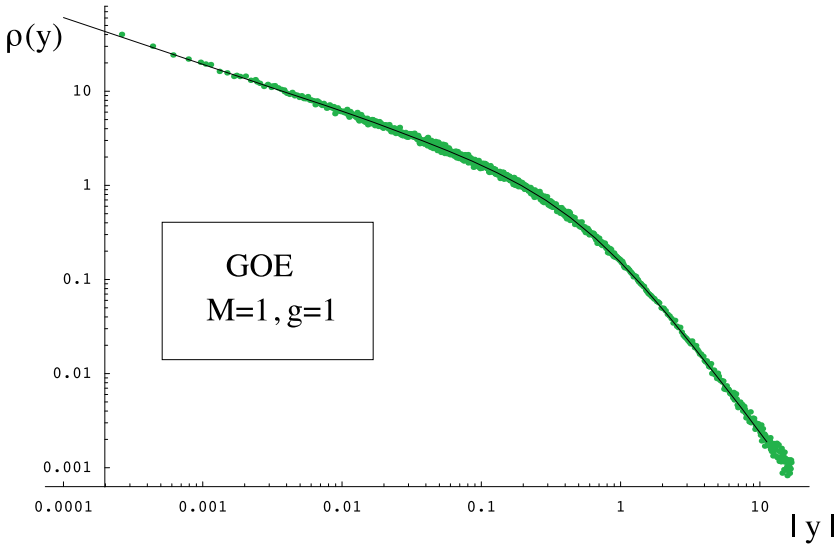

Figure 1. Comparison between the predictions of our formula (4) for one channel case $M = 1$ at 'perfect coupling' $g = 1$ and the results of direct numerical diagonalization of $K = 10\,000$ complex matrices of the size 400×400.

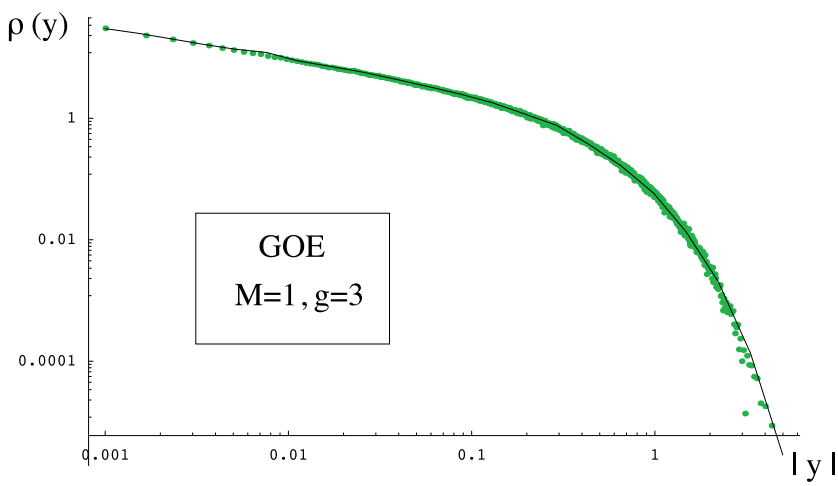

Figure 2. Same as figure 1 for $M = 1$, $g = 3$.

which coincides with the distribution derived recently by Alhassid and Lewenkopf [13] by a perturbative treatment, up to alternating signs in the factor $\chi_2(p)$ which is an apparent misprint in their paper.

Relying on earlier experience with resonance statistics in systems with broken time-reversal invariance [11, 16], one might expect that the distribution simplifies drastically for the case of statistically equivalent channels: $g \equiv g_1 = g_2 = \cdots = g_M$. As a first step toward this limit it is natural to consider the limit of even number $M = 2M_{eff}$ of 'pairwise-equivalent' channels: $g_1 = g_2 \equiv \tilde{g}_1$, $g_3 = g_4 \equiv \tilde{g}_2, \ldots, g_{2M_{eff}-1} = g_{2M_{eff}} \equiv \tilde{g}_{M_{eff}}$. In this limit the factors $\chi_{1,2}(p)$ drop out from the integrand and the general expression (3) can

indeed be reduced to a somewhat simplier form amounting to replacing the function $F(\lambda, y)$ in equation (4) with $F_{M_{eff}}$ given for $M_{eff} = 1$ by

$$F|_{M_{eff}=1} = \pi(\tilde{g} - \lambda)^2 f(\tilde{g}) \int_1^{\tilde{g}} \mathrm{d}p\, f(p) \qquad \text{where} \quad f(p) = \frac{1}{(p^2-1)^{1/2}} \frac{\mathrm{e}^{yp}}{(\lambda - p)^2} \tag{6}$$

and for $M_{eff} \geqslant 2$ by

$$F|_{M_{eff}} = \pi \prod_a^{M_{eff}} (\tilde{g}_a - \lambda)^2 \sum_{a=1}^{M_{eff}} \frac{f(\tilde{g}_a)}{\prod_{b\neq a}(\tilde{g}_b - \tilde{g}_a)} \sum_{c\neq a}^{M_{eff}} \frac{1}{\prod_{q\neq a,c}(\tilde{g}_q - \tilde{g}_c)} \times \left[\int_1^{\tilde{g}_a} \mathrm{d}p\, \frac{f(p) - f(\tilde{g}_c)}{\tilde{g}_c - p} - f(\tilde{g}_c) \ln(\tilde{g}_c - 1) \right]. \tag{7}$$

One can use equation (7) for explicitly performing the transition to the case of all-equivalent channels. Surprisingly, the general expression turns out to be rather awkward even for the simplest case of four equivalent channels, $M_{eff} = 2$. For this reason we do not give the corresponding formula here.

Despite the fact, that the general resonance widths distribution for the case of preserved TRS turns out to be much more complicated than for the case of broken TRS, both expressions share many important features. For example, evaluating the first moment of the distribution equations (4), (6) for $M_{eff} = 1$ exactly, one arrives at the following expression for the mean resonance widths:

$$\frac{\langle \Gamma \rangle}{\Delta} = -\frac{1}{\pi} \ln \frac{g-1}{g+1}. \tag{8}$$

This formula is well known in nuclear physics as the Moldauer–Simonius relation and follows from the unitarity of the scattering matrix [30]. Actually, a simlar relation must be satisfied for an arbitrary number M of open channels. It was indeed found to be the case for systems with broken TRS [11]. Unfortunately, it is quite difficult to directly verify the Moldauer–Simonius relation from the general distribution equation (3).

The logarithmic divergency at $g_a = 1$ signals on the power-law tail behaviour $\rho_X(|y| \gg 1) \propto y^{-2}$ typical for this (a so-called 'perfect coupling') case. It is instructive to perform the limit $g \to 1$ directly in equation (6) for $M_{eff} = 1$ and find:

$$\rho_X(y < 0) = \frac{1}{4y^3}(2y + 1 + (2y - 1)\mathrm{e}^{4y}) \tag{9}$$

where such a behaviour is self-evident.

Actually, it is interesting to compare this expression with the $M = 1$ case for systems with broken TRS [16]:

$$\rho_X(y < 0) = \frac{1}{2y^2}(1 + (2y - 1)\mathrm{e}^{2y}). \tag{10}$$

The reason is that in the regime of a few weakly open equivalent channels (isolated resonances), $g \gg 1$, the M-channel system with broken TRS possesses exactly the same widths distribution as the $2M$ channel system with unbroken TRS. A similar correspondence also holds in the limit of very many equivalent open channels, $M \gg g$, where one can show that (see [11, 15]):

$$\rho(y < 0)_{M\gg g} = \begin{cases} \dfrac{M_{eff}}{2y^2} & -\dfrac{M_{eff}}{g+1} > y > -\dfrac{M_{eff}}{g-1} \\ 0 & y < -\dfrac{M_{eff}}{g-1} \quad \text{or} \quad y > -\dfrac{M_{eff}}{g+1} \end{cases} \tag{11}$$

with $M_{eff} = M$ for broken and $M_{eff} = M/2$ for unbroken TRS.

Moreover, before the exact formula equation (9) was available, an attempt was made [31] to compare the $M = 1$ formula (10) valid for systems with broken TRS with numerically found resonance widths in an $M = 2$ chaotic TRS-preserving microwave cavity. The correspondence was reported to be quite satisfactory.

We see that both formulae (9), (10) indeed give exactly the same behaviour for the resonances whose widths exceed the typical separation between them: $\rho_X(|y| \gg 1) \approx 1/2y^2$. However, in the region of very narrow resonances the behaviour is slightly different: $\rho_X(y \to 0) = 1$ for broken and $\rho(y \to 0) = \frac{4}{3}$ for unbroken TRS. This fact shows the limitations of the mentioned correspondence.

Discussing the possibility of a verification of our expressions on realistic models of quantum chaotic systems, such as e.g. a three-disc scattering considered in [12], it is important to have in mind that it is necessary to keep the number M of open channels to be of the order of unity. This was not the case for the studies undertaken by Gaspard and collaborators [12], where the number of open channels was typically very large. As a result, no resonances with small widths were detected, in qualitative agreement with the limiting distribution equation (11). To go beyond this limit, it is necessary to consider the separation between discs to be small in comparison with the disc radius. A recent paper [32] discusses such a possibility.

As to the random-perturbation case ($s = R$ in equation (2)), the presence of the Gaussian factors in the integrand of equation (2) prevents one from deforming the contour to the complex plane. At the moment, we are unable to simplify the expression further. However, one can still infer that in the perturbative case, $v \ll 1$, the function $\rho_X(y)$ reduces to a simple Gaussian, whereas in the opposite limiting case, $v \gg 1$, one indeed arrives at the 'elliptic law' well known to hold for strongly non-hermitian random matrices [26, 27].

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